2.3 Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

In Section 2.2, you learned that the derivative of the sum of two functions is simply the sum of their derivatives. The rules for the derivatives of the product and quotient of two functions are not as simple.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

REMARK A version of the Product Rule that some people prefer is

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

The advantage of this form is that it generalizes easily to products of three or more factors.

Proof Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may appear unmotivated to a reader. This proof involves such a step—subtracting and adding the same quantity—which is shown in color.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]\\
&= \lim_{\Delta x\to 0} f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} + \lim_{\Delta x\to 0} g(x)\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Note that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$ because f is given to be differentiable and therefore is continuous.

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

REMARK The proof of the Product Rule for products of more than two factors is left as an exercise (see Exercise 137).

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

So, the derivative of $y = x^2 \sin x \cos x$ is

$$\begin{aligned}
\frac{dy}{dx} &= 2x\sin x\cos x + x^2\cos x\cos x + x^2\sin x(-\sin x)\\
&= 2x\sin x\cos x + x^2(\cos^2 x - \sin^2 x).
\end{aligned}$$

THE PRODUCT RULE

When Leibniz originally wrote a formula for the Product Rule, he was motivated by the expression

$$(x + dx)(y + dy) - xy$$

from which he subtracted $dx\,dy$ (as being negligible) and obtained the differential form $x\,dy + y\,dx$. This derivation resulted in the traditional form of the Product Rule. *(Source: The History of Mathematics by David M. Burton)*

The derivative of a product of two functions is not (in general) given by the product of the derivatives of the two functions. To see this, try comparing the product of the derivatives of

$$f(x) = 3x - 2x^2$$

and

$$g(x) = 5 + 4x$$

with the derivative in Example 1.

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$\begin{aligned} h'(x) &= \underbrace{(3x - 2x^2)}_{\text{First}}\underbrace{\frac{d}{dx}[5 + 4x]}_{\text{Derivative of second}} + \underbrace{(5 + 4x)}_{\text{Second}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{\text{Derivative of first}} && \text{Apply Product Rule.} \\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\ &= (12x - 8x^2) + (15 - 8x - 16x^2) \\ &= -24x^2 + 4x + 15 \end{aligned}$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$\begin{aligned} D_x[(3x - 2x^2)(5 + 4x)] &= D_x[-8x^3 + 2x^2 + 15x] \\ &= -24x^2 + 4x + 15. \end{aligned}$$

In the next example, you must use the Product Rule.

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\begin{aligned} \frac{d}{dx}[3x^2 \sin x] &= 3x^2\frac{d}{dx}[\sin x] + \sin x\frac{d}{dx}[3x^2] && \text{Apply Product Rule.} \\ &= 3x^2 \cos x + (\sin x)(6x) \\ &= 3x^2 \cos x + 6x \sin x \\ &= 3x(x \cos x + 2 \sin x) \end{aligned}$$

REMARK In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

EXAMPLE 3 Using the Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \underbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}_{\text{Product Rule}} - \underbrace{2\frac{d}{dx}[\sin x]}_{\text{Constant Multiple Rule}} \\ &= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x) \\ &= -2x \sin x \end{aligned}$$

The Quotient Rule

THEOREM 2.8 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

REMARK From the Quotient Rule, you can see that the derivative of a quotient is not (in general) the quotient of the derivatives.

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$= \frac{\lim\limits_{\Delta x \to 0} \dfrac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim\limits_{\Delta x \to 0} \dfrac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\lim\limits_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

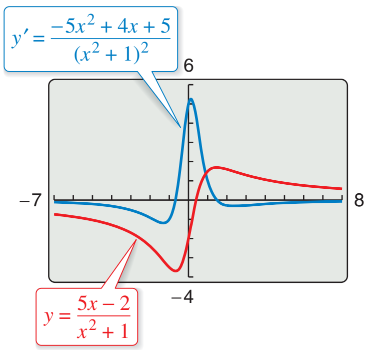
$$= \frac{g(x)\left[\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\lim\limits_{\Delta x \to 0} \dfrac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\lim\limits_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Note that $\lim\limits_{\Delta x \to 0} g(x + \Delta x) = g(x)$ because g is given to be differentiable and therefore is continuous.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

TECHNOLOGY A graphing utility can be used to compare the graph of a function with the graph of its derivative. For instance, in Figure 2.22, the graph of the function in Example 4 appears to have two points that have horizontal tangent lines. What are the values of y' at these two points?



Graphical comparison of a function and its derivative
Figure 2.22

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x - 2}{x^2 + 1}$.

Solution

$$\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] = \frac{(x^2+1)\frac{d}{dx}[5x-2] - (5x-2)\frac{d}{dx}[x^2+1]}{(x^2+1)^2} \qquad \text{Apply Quotient Rule.}$$

$$= \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2}$$

$$= \frac{(5x^2+5) - (10x^2 - 4x)}{(x^2+1)^2}$$

$$= \frac{-5x^2 + 4x + 5}{(x^2+1)^2}$$

Note the use of parentheses in Example 4. A liberal use of parentheses is recommended for *all* types of differentiation problems. For instance, with the Quotient Rule, it is a good idea to enclose all factors and derivatives in parentheses, and to pay special attention to the subtraction required in the numerator.

When differentiation rules were introduced in the preceding section, the need for rewriting *before* differentiating was emphasized. The next example illustrates this point with the Quotient Rule.

EXAMPLE 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x) = \dfrac{3 - (1/x)}{x + 5}$ at $(-1, 1)$.

Solution Begin by rewriting the function.

$$f(x) = \frac{3 - (1/x)}{x + 5} \qquad \text{Write original function.}$$

$$= \frac{x\left(3 - \frac{1}{x}\right)}{x(x + 5)} \qquad \text{Multiply numerator and denominator by } x.$$

$$= \frac{3x - 1}{x^2 + 5x} \qquad \text{Rewrite.}$$

Next, apply the Quotient Rule.

$$f'(x) = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2} \qquad \text{Quotient Rule}$$

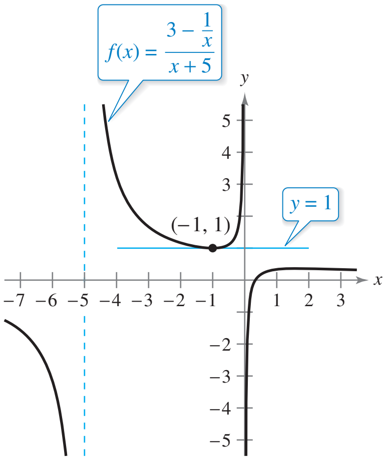
$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2} \qquad \text{Simplify.}$$

To find the slope at $(-1, 1)$, evaluate $f'(-1)$.

$$f'(-1) = 0 \qquad \text{Slope of graph at } (-1, 1)$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at $(-1, 1)$ is $y = 1$. See Figure 2.23.



The line $y = 1$ is tangent to the graph of $f(x)$ at the point $(-1, 1)$.
Figure 2.23

Not every quotient needs to be differentiated by the Quotient Rule. For instance, each quotient in the next example can be considered as the product of a constant times a function of x. In such cases, it is more convenient to use the Constant Multiple Rule.

REMARK To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

EXAMPLE 6 Using the Constant Multiple Rule

	Original Function	Rewrite	Differentiate	Simplify
a.	$y = \dfrac{x^2 + 3x}{6}$	$y = \dfrac{1}{6}(x^2 + 3x)$	$y' = \dfrac{1}{6}(2x + 3)$	$y' = \dfrac{2x + 3}{6}$
b.	$y = \dfrac{5x^4}{8}$	$y = \dfrac{5}{8}x^4$	$y' = \dfrac{5}{8}(4x^3)$	$y' = \dfrac{5}{2}x^3$
c.	$y = \dfrac{-3(3x - 2x^2)}{7x}$	$y = -\dfrac{3}{7}(3 - 2x)$	$y' = -\dfrac{3}{7}(-2)$	$y' = \dfrac{6}{7}$
d.	$y = \dfrac{9}{5x^2}$	$y = \dfrac{9}{5}(x^{-2})$	$y' = \dfrac{9}{5}(-2x^{-3})$	$y' = -\dfrac{18}{5x^3}$

In Section 2.2, the Power Rule was proved only for the case in which the exponent n is a positive integer greater than 1. The next example extends the proof to include negative integer exponents.

EXAMPLE 7 Power Rule: Negative Integer Exponents

If n is a negative integer, then there exists a positive integer k such that $n = -k$. So, by the Quotient Rule, you can write

$$\frac{d}{dx}[x^n] = \frac{d}{dx}\left[\frac{1}{x^k}\right]$$

$$= \frac{x^k(0) - (1)(kx^{k-1})}{(x^k)^2} \qquad \text{Quotient Rule and Power Rule}$$

$$= \frac{0 - kx^{k-1}}{x^{2k}}$$

$$= -kx^{-k-1}$$

$$= nx^{n-1}. \qquad n = -k$$

So, the Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 71 in Section 2.5, you are asked to prove the case for which n is any rational number.

Derivatives of Trigonometric Functions

Knowing the derivatives of the sine and cosine functions, you can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

REMARK In the proof of Theorem 2.9, note the use of the trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

and

$$\sec x = \frac{1}{\cos x}.$$

These trigonometric identities and others are listed in Appendix C and on the formula cards for this text.

Proof Considering $\tan x = (\sin x)/(\cos x)$ and applying the Quotient Rule, you obtain

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \qquad \text{Apply Quotient Rule.}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The proofs of the other three parts of the theorem are left as an exercise (see Exercise 87).

EXAMPLE 8 Differentiating Trigonometric Functions

See LarsonCalculus.com for an interactive version of this type of example.

Function	**Derivative**
a. $y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x \tan x)$

EXAMPLE 9 Different Forms of a Derivative

REMARK Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

Differentiate both forms of

$$y = \frac{1 - \cos x}{\sin x} = \csc x - \cot x.$$

Solution

First form:
$$y = \frac{1 - \cos x}{\sin x}$$
$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$
$$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$
$$= \frac{1 - \cos x}{\sin^2 x} \qquad \sin^2 x + \cos^2 x = 1$$

Second form:
$$y = \csc x - \cot x$$
$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$
$$= \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$
$$= \csc^2 x - \csc x \cot x.$$

The summary below shows that much of the work in obtaining a simplified form of a derivative occurs *after* differentiating. Note that two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

	$f'(x)$ After Differentiating	$f'(x)$ After Simplifying
Example 1	$(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$	$-24x^2 + 4x + 15$
Example 3	$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$	$-2x \sin x$
Example 4	$\dfrac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$	$\dfrac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$
Example 5	$\dfrac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$	$\dfrac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$
Example 6	$\dfrac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$	$\dfrac{1 - \cos x}{\sin^2 x}$

Higher-Order Derivatives

Just as you can obtain a velocity function by differentiating a position function, you can obtain an **acceleration** function by differentiating a velocity function. Another way of looking at this is that you can obtain an acceleration function by differentiating a position function *twice*.

$$s(t) \qquad \text{Position function}$$
$$v(t) = s'(t) \qquad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \qquad \text{Acceleration function}$$

The function $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

The second derivative is an example of a **higher-order derivative.** You can define derivatives of any positive integer order. For instance, the **third derivative** is the derivative of the second derivative. Higher-order derivatives are denoted as shown below.

First derivative: $y', \quad f'(x), \quad \dfrac{dy}{dx}, \quad \dfrac{d}{dx}[f(x)], \quad D_x[y]$

Second derivative: $y'', \quad f''(x), \quad \dfrac{d^2y}{dx^2}, \quad \dfrac{d^2}{dx^2}[f(x)], \quad D_x^2[y]$

Third derivative: $y''', \quad f'''(x), \quad \dfrac{d^3y}{dx^3}, \quad \dfrac{d^3}{dx^3}[f(x)], \quad D_x^3[y]$

Fourth derivative: $y^{(4)}, \quad f^{(4)}(x), \quad \dfrac{d^4y}{dx^4}, \quad \dfrac{d^4}{dx^4}[f(x)], \quad D_x^4[y]$

$\vdots$

nth derivative: $y^{(n)}, \quad f^{(n)}(x), \quad \dfrac{d^ny}{dx^n}, \quad \dfrac{d^n}{dx^n}[f(x)], \quad D_x^n[y]$

REMARK The second derivative of a function is the derivative of the first derivative of the function.

The moon's mass is 7.349×10^{22} kilograms, and Earth's mass is 5.976×10^{24} kilograms. The moon's radius is 1737 kilometers, and Earth's radius is 6378 kilometers. Because the gravitational force on the surface of a planet is directly proportional to its mass and inversely proportional to the square of its radius, the ratio of the gravitational force on Earth to the gravitational force on the moon is

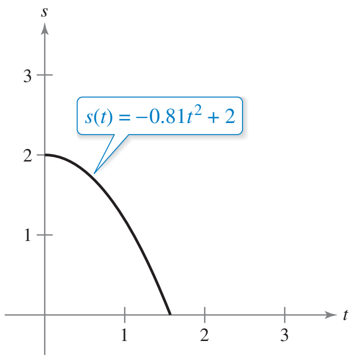
$$\frac{(5.976 \times 10^{24})/6378^2}{(7.349 \times 10^{22})/1737^2} \approx 6.0.$$

EXAMPLE 10 **Finding the Acceleration Due to Gravity**

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds, as shown in the figure at the right. What is the ratio of Earth's gravitational force to the moon's?



Solution To find the acceleration, differentiate the position function twice.

$$s(t) = -0.81t^2 + 2 \qquad \text{Position function}$$
$$s'(t) = -1.62t \qquad \text{Velocity function}$$
$$s''(t) = -1.62 \qquad \text{Acceleration function}$$

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.0.$$

NASA

2.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using the Product Rule In Exercises 1–6, use the Product Rule to find the derivative of the function.

1. $g(x) = (x^2 + 3)(x^2 - 4x)$
2. $y = (3x - 4)(x^3 + 5)$
3. $h(t) = \sqrt{t}(1 - t^2)$
4. $g(s) = \sqrt{s}(s^2 + 8)$
5. $f(x) = x^3 \cos x$
6. $g(x) = \sqrt{x} \sin x$

Using the Quotient Rule In Exercises 7–12, use the Quotient Rule to find the derivative of the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $g(t) = \dfrac{3t^2 - 1}{2t + 5}$
9. $h(x) = \dfrac{\sqrt{x}}{x^3 + 1}$
10. $f(x) = \dfrac{x^2}{2\sqrt{x} + 1}$
11. $g(x) = \dfrac{\sin x}{x^2}$
12. $f(t) = \dfrac{\cos t}{t^3}$

Finding and Evaluating a Derivative In Exercises 13–18, find $f'(x)$ and $f'(c)$.

Function	Value of c
13. $f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$	$c = 0$
14. $y = (x^2 - 3x + 2)(x^3 + 1)$	$c = 2$
15. $f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
16. $f(x) = \dfrac{x - 4}{x + 4}$	$c = 3$
17. $f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
18. $f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

Using the Constant Multiple Rule In Exercises 19–24, complete the table to find the derivative of the function without using the Quotient Rule.

Function	Rewrite	Differentiate	Simplify
19. $y = \dfrac{x^2 + 3x}{7}$			
20. $y = \dfrac{5x^2 - 3}{4}$			
21. $y = \dfrac{6}{7x^2}$			
22. $y = \dfrac{10}{3x^3}$			
23. $y = \dfrac{4x^{3/2}}{x}$			
24. $y = \dfrac{2x}{x^{1/3}}$			

Finding a Derivative In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$
26. $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 4}$
27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
29. $f(x) = \dfrac{3x - 1}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 + 3)^3$
33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
35. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$
36. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$
37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant
38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

Finding a Derivative of a Trigonometric Function In Exercises 39–54, find the derivative of the trigonometric function.

39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \dfrac{\cos t}{t}$
42. $f(x) = \dfrac{\sin x}{x^3}$
43. $f(x) = -x + \tan x$
44. $y = x + \cot x$
45. $g(t) = \sqrt[4]{t} + 6 \csc t$
46. $h(x) = \dfrac{1}{x} - 12 \sec x$
47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
48. $y = \dfrac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \sin x + \cos x$
51. $f(x) = x^2 \tan x$
52. $f(x) = \sin x \cos x$
53. $y = 2x \sin x + x^2 \cos x$
54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

Finding a Derivative Using Technology In Exercises 55–58, use a computer algebra system to find the derivative of the function.

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$
57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$
58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

Evaluating a Derivative **In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.**

Function	Point
59. $y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60. $f(x) = \tan x \cot x$	$(1, 1)$
61. $h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62. $f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

Finding an Equation of a Tangent Line **In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.**

63. $f(x) = (x^3 + 4x - 1)(x - 2),\quad (1, -4)$

64. $f(x) = (x - 2)(x^2 + 4),\quad (1, -5)$

65. $f(x) = \dfrac{x}{x + 4},\quad (-5, 5)$

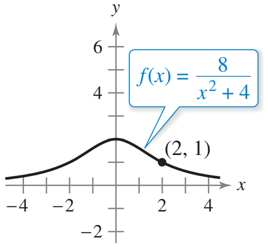
66. $f(x) = \dfrac{x + 3}{x - 3},\quad (4, 7)$

67. $f(x) = \tan x,\quad \left(\dfrac{\pi}{4}, 1\right)$

68. $f(x) = \sec x,\quad \left(\dfrac{\pi}{3}, 2\right)$

Famous Curves **In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *Witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)**

69.



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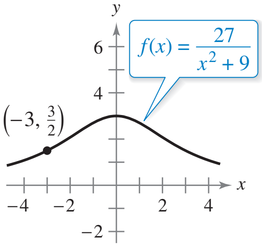

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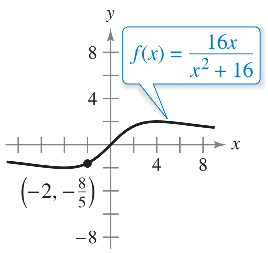

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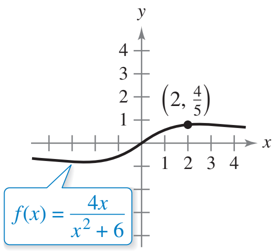


Horizontal Tangent Line **In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.**

73. $f(x) = \dfrac{2x - 1}{x^2}$

74. $f(x) = \dfrac{x^2}{x^2 + 1}$

75. $f(x) = \dfrac{x^2}{x - 1}$

76. $f(x) = \dfrac{x - 4}{x^2 - 7}$

77. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = (x + 1)/(x - 1)$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

78. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = x/(x - 1)$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

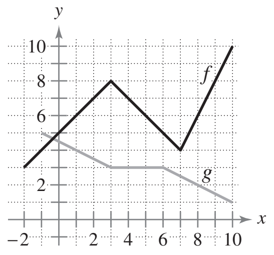
Exploring a Relationship **In Exercises 79 and 80, verify that $f'(x) = g'(x)$, and explain the relationship between f and g.**

79. $f(x) = \dfrac{3x}{x + 2},\quad g(x) = \dfrac{5x + 4}{x + 2}$

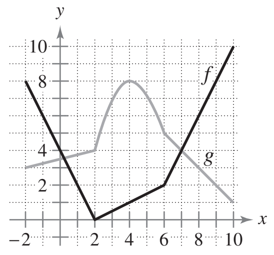
80. $f(x) = \dfrac{\sin x - 3x}{x},\quad g(x) = \dfrac{\sin x + 2x}{x}$

Evaluating Derivatives **In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.**

81. (a) Find $p'(1)$.
(b) Find $q'(4)$.



82. (a) Find $p'(4)$.
(b) Find $q'(7)$.



83. Area The length of a rectangle is given by $6t + 5$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

84. Volume The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

85. Inventory Replenishment The ordering and transportation cost C for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right),\quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

86. Population Growth A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

87. Proof Prove the following differentiation rules.

(a) $\dfrac{d}{dx}[\sec x] = \sec x \tan x$

(b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\dfrac{d}{dx}[\cot x] = -\csc^2 x$

88. Rate of Change Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

89. Modeling Data The table shows the health care expenditures h (in billions of dollars) in the United States and the population p (in millions) of the United States for the years 2004 through 2009. The year is represented by t, with $t = 4$ corresponding to 2004. *(Source: U.S. Centers for Medicare & Medicaid Services and U.S. Census Bureau)*

Year, t	4	5	6	7	8	9
h	1773	1890	2017	2135	2234	2330
p	293	296	299	302	305	307

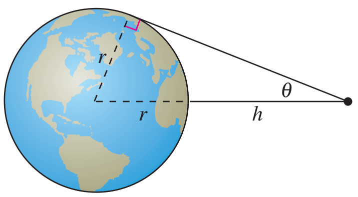
(a) Use a graphing utility to find linear models for the health care expenditures $h(t)$ and the population $p(t)$.

(b) Use a graphing utility to graph each model found in part (a).

(c) Find $A = h(t)/p(t)$, then graph A using a graphing utility. What does this function represent?

(d) Find and interpret $A'(t)$ in the context of these data.

90. Satellites When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface, and let r represent Earth's radius.



(a) Show that $h = r(\csc \theta - 1)$.

(b) Find the rate at which h is changing with respect to θ when $\theta = 30°$. (Assume $r = 3960$ miles.)

Finding a Second Derivative **In Exercises 91–98, find the second derivative of the function.**

91. $f(x) = x^4 + 2x^3 - 3x^2 - x$

92. $f(x) = 4x^5 - 2x^3 + 5x^2$

93. $f(x) = 4x^{3/2}$

94. $f(x) = x^2 + 3x^{-3}$

95. $f(x) = \dfrac{x}{x-1}$

96. $f(x) = \dfrac{x^2 + 3x}{x - 4}$

97. $f(x) = x \sin x$

98. $f(x) = \sec x$

Finding a Higher-Order Derivative **In Exercises 99–102, find the given higher-order derivative.**

99. $f'(x) = x^2, \quad f''(x)$

100. $f''(x) = 2 - \dfrac{2}{x}, \quad f'''(x)$

101. $f'''(x) = 2\sqrt{x}, \quad f^{(4)}(x)$

102. $f^{(4)}(x) = 2x + 1, \quad f^{(6)}(x)$

Using Relationships **In Exercises 103–106, use the given information to find $f'(2)$.**

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

103. $f(x) = 2g(x) + h(x)$

104. $f(x) = 4 - h(x)$

105. $f(x) = \dfrac{g(x)}{h(x)}$

106. $f(x) = g(x)h(x)$

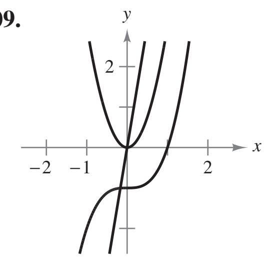
WRITING ABOUT CONCEPTS

107. Sketching a Graph Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$. Explain how you found your answer.

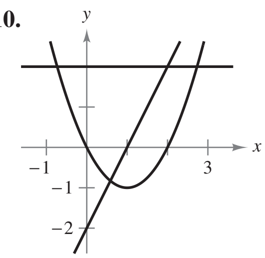
108. Sketching a Graph Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x. Explain how you found your answer.

Identifying Graphs **In Exercises 109 and 110, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

109.

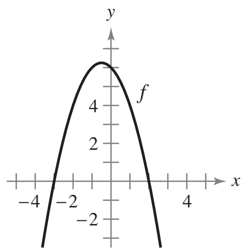


110.



Sketching Graphs **In Exercises 111–114, the graph of f is shown. Sketch the graphs of f' and f''. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

111.



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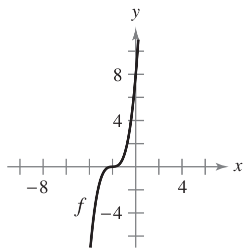

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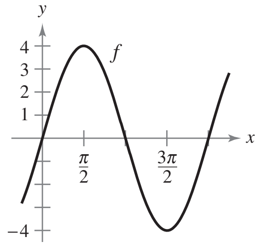
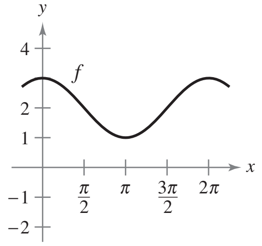


114.



115. Acceleration The velocity of an object in meters per second is

$$v(t) = 36 - t^2$$

for $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

116. Acceleration The velocity of an automobile starting from rest is

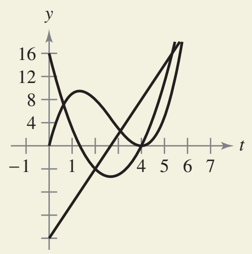
$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at (a) 5 seconds, (b) 10 seconds, and (c) 20 seconds.

117. Stopping Distance A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is $s(t) = -8.25t^2 + 66t$, where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

118. HOW DO YOU SEE IT? The figure shows the graphs of the position, velocity, and acceleration functions of a particle.



(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.

Finding a Pattern In Exercises 119 and 120, develop a general rule for $f^{(n)}(x)$ given $f(x)$.

119. $f(x) = x^n$ **120.** $f(x) = \dfrac{1}{x}$

121. Finding a Pattern Consider the function $f(x) = g(x)h(x)$.

(a) Use the Product Rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results of part (a) to write a general rule for $f^{(n)}(x)$.

122. Finding a Pattern Develop a general rule for $[xf(x)]^{(n)}$, where f is a differentiable function of x.

Finding a Pattern In Exercises 123 and 124, find the derivatives of the function f for $n = 1, 2, 3,$ and 4. Use the results to write a general rule for $f'(x)$ in terms of n.

123. $f(x) = x^n \sin x$ **124.** $f(x) = \dfrac{\cos x}{x^n}$

Differential Equations In Exercises 125–128, verify that the function satisfies the differential equation.

	Function	Differential Equation
125.	$y = \dfrac{1}{x}, x > 0$	$x^3 y'' + 2x^2 y' = 0$
126.	$y = 2x^3 - 6x + 10$	$-y''' - xy'' - 2y' = -24x^2$
127.	$y = 2 \sin x + 3$	$y'' + y = 3$
128.	$y = 3 \cos x + \sin x$	$y'' + y = 0$

True or False? In Exercises 129–134, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

129. If $y = f(x)g(x)$, then $\dfrac{dy}{dx} = f'(x)g'(x)$.

130. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $\dfrac{d^5y}{dx^5} = 0$.

131. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

132. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

133. The second derivative represents the rate of change of the first derivative.

134. If the velocity of an object is constant, then its acceleration is zero.

135. Absolute Value Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist? (*Hint:* Rewrite the function as a piecewise function and then differentiate each part.)

136. Think About It Let f and g be functions whose first and second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$ (b) $fg'' + f''g = (fg)''$

137. Proof Use the Product Rule twice to prove that if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$